PLANE STRAIN SHAKEDOWN ANALYSIS

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Résumé :

The finite element method is coupled with linear programming to determine shakedown load factors for elastic perfectly plastic embankment subjected to repeated loading using the static shakedown theorem.

Abstract:

Shakedown load factors are evaluated for elastic perfectly plastic embankment subjected to repeated loading using the static shakedown theorem. The finite element method is coupled with linear programming to determine optimal loads. The discretized continuum is modeled as an elastic-perfectly plastic material with piecewise linearized Mohr-Coulomb yield surface and its associated flow rule. The results are obtained using special purpose finite element-linear programming code developed earlier to determine shakedown and limit load factors and associated stresses and deformations. In this paper, application to field problems associated with highway engineering is presented. The results presented can be used to benchmark numerical shakedown results, as well as to serve as a theoretical basis for the development of an analytical design method for roads under moving traffic loads.

1. INTRODUCTION:

Roads are usually exposed to variable repeated loading due to the continuous vehicle loading application with different wheel load magnitudes. Both load magnitudes and number of repetitions of load need to be considered in order to avoid significant damages to roads. Conventional design procedures assume that roads deteriorate indefinitely. However, this is not always true as steady-state shakedown conditions are frequently achieved. Shakedown safety factor is one of the most important items to be checked in the design of such continua. In general, the methods of determining the safety of structures, based on the theory of plasticity, are characterized by the types of loading. For loads varying according to a prescribed time history, it is possible to follow the development of deformations through step-by-step integration of the elastoplastic equations up to the collapse. For loads increasing proportionally, with the assumption of perfect plasticity, the theorems of limit analysis can provide the collapse load and mechanism either by a trial and error procedure, or by mathematical programming. However, if a structure is subjected to cyclic loads varying between prescribed limits, these theorems will give unsafe estimate of the collapse loads as failure will occur at loads well below the static collapse values. Shakedown theorems which are the quasi-static generalization of limit theorems can provide appropriate safe bounds for those loads. Depending on the nature of the cyclic plastic deformations produced during the continuous cycling, shakedown analysis discerns safe and failure states as follows: The safe state occurs when plastic deformation stabilizes after a finite number of load cycles after which the continuum responds to any further loading cycles in a purely elastic manner. On the other hand, failure state occurs when plastic deformation does not stabilize, and hence the structure becomes unserviceable either due to alternating plasticity and/or incremental collapse. Alternating plasticity is a local mode of failure due to repeated development of plastic strains in an alternating sense and in a bounded form at a certain point in the continuum. Incremental collapse is a global mode of failure due to accumulation of plastic strains of the same sense at the end of each load cycle leading to progressive increase of permanent deformation. It is worth noting that for the particular case of proportional loading, the incremental collapse mode reduces to the collapse mechanism of limit analysis. To save the computational effort in shakedown and limit analyses, the nonlinear formulations of the static and kinematic theorems are adapted to linear programming by piecewise linearization of the yield surfaces. The duality property of linear programming is used to relate the static formulation to the kinematic one. As a direct calculation of the exact shakedown load is difficult, it is necessary to estimate the optimum lower or upper bounds to the shakedown limits through finite element and linear programming. The finite element method is coupled with linear programming techniques to determine the optimum shakedown loads. The elastic perfectly plastic piecewise linear constitutive law for each element is defined using the piecewise linear Mohr-Coulomb yield criterion along with its associated flow rule.

Melan [1] and Koiter [2] formulated, respectively the static and kinematic shakedown theorems for elastic perfectly-plastic materials. Maier [3] adapted the non-linear formulation of these two theorems into linear programming by piecewise linearization of the yield surfaces. The finite element discretization and linear programming were used to formulate a general matrix shakedown theory, in which, the shakedown analysis of continuum problems is restated as two dual linear programming problems. Belytschko [4] formulated Melan's theorem as a nonlinear programming problem. An equilibrated finite element was used to analyze the shakedown of a square plate of an elastic-perfectly plastic "von Mises" material with a central circular hole subjected to biaxial variable repeated loading under plane stress condition. Corradi and Zavelani [5] modified Maier's approach to Melan's theorem to analyze the same plane stress problem solved earlier by Belyschko [4], but with a compatible finite element model and piecewise linearized von Mises yield criterion. Hung and Palgen [6] reduced the number of nonlinear yield constraints in Belytschko's formulation [4] by averaging von Mises yield function over each element. The method was applied to solve the same plane stress problem solved earlier by Belytschko [4]. Aboustit and Reddy [7] used Maier's approach to Koiter's Kinematic shakedown theory to solve Belytschko' problem [4]. The application was also illustrated for a plane strain footing resting on Mohr-Coulomb soil and subjected to eccentric inclined variable repeated loading.

Using constant strain triangular element to evaluate optimal loads for plane strain continua, Aboustait [8] showed that the static formulations for both shakedown and limit analyses, provides more accurate estimate of the collapse load and in some instances were found to consume fewer iterations than the kinematic formulations. Jeragh and Aboustait [9] ran experimental shakedown analysis on sand with different degrees of compaction subjected to cyclic loading showed that either the sand will be stabilized after a sufficiently large number of cycles to a 'nearly elastic material' and approaching a safe state or it fails after a small number of cycles by cyclic plastic deformations in an incremental collapse mode. Applications of shakedown analysis to road pavement design were presented by Shiau et al [10] and Ravindra et al [11].

2. FORMULATIONS:

The present study is an application of Maier's quasi-static matrix shakedown theory [3]. This theory neglects the effect of change in the configuration on the equilibrium equations by assuming small strain hypothesis and it also neglects both inertia and damping effects by assuming that loads are acting sufficiently slowly. Basically this theory requires the following:-

1) Discretizing the continuum into constant strain triangular elements. The yield conditions being checked only at a finite number of 'check points' coinciding with the element centeroids, and

2) Piecewise linearizing the yield surface at every check point i, into a polyhedron of Yi planes defined by unit normal vectors assembled in the matrix $[N^i]$ and the normal distances assembled in the vector $K^i o$ from the stress origin to each yield plane. If σ^i is the total stress vector at a check point i, then the yield condition at that point can be written as:

$$[\mathbf{N}^i]^T \boldsymbol{\sigma}^i \leq \mathbf{K}^i\mathbf{o} \qquad (1)$$

The stress distribution under cyclic loading in the inelastic range can be expressed as the sum of an elastic solution (fictitious stress, assuming the material has unlimited elastic behavior), σ^E, and a self-equilibrated residual stress distribution, σ^R. In this case, the yield condition for the global system is

$$[\mathbf{N}]^T (\boldsymbol{\sigma}^E(\mathbf{x},t) + \boldsymbol{\sigma}^R(\mathbf{x},\mathbf{t})) \leq \mathbf{Ko} \quad (2)$$

2.1 Melan's Theorem:

The static shakedown theorem, known as Melan's theorem, basically states that, an elastic-perfectly plastic structure when subjected to loads F(t), varying within a prescribed limit with unknown history, will shake down under these loads if a time-independent residual stress, $\sigma^R(x)$ can be found such that:

$$[\mathbf{N}]^T (\boldsymbol{\sigma}^E(\mathbf{x},t) + \boldsymbol{\sigma}^R(\mathbf{x})) \leq \mathbf{Ko} \quad (3)$$

for every x and t; where $\sigma^E(x,t)$ is the linear elastic response of the structure to F(t).

The theorem provides the shakedown load factor as the maximum of all statically admissible multipliers, k, giving rise to the following linear programming problem,

$$s = \max_{k,\, \boldsymbol{\sigma}^R} k,$$

subjected to

a) $[\mathbf{E}]^T \boldsymbol{\sigma}^R = \mathbf{0}$,

b) $k\,\mathbf{M} + [\mathbf{N}]^T \boldsymbol{\sigma}^R \leq \mathbf{Ko}$,

c) $k \geq 0$ (4)

Where M is the maximum with respect to time of the projection of the global elastic stress vector $\sigma^E(x,t)$ on the normality matrix [N]. The constraints identified at Eq(4a) represent the self equilibrated residual stress condition, where $[E]^T$ is the global equilibrium matrix. The number of equations represented by (4a) equals the number of degrees of freedom, and those corresponding to constrained displacements have to be eliminated. Equation (4) is a quasi-static generalization of the finite element-linear programming approach to the static theorem of limit analysis, Anderheggen [12].

2.2 Koiter's Theorem:

In linear programming theory, any maximization (minimization) problem can be associated with a minimization (maximization) problem having the same vectors and matrices and connected with the former by some duality properties Pierre [13]. By the dualization, the dual program can be obtained from the primal, and will be:

$$s = \min_{\mathbf{q},\, \boldsymbol{\lambda}} \mathbf{Ko}^T \boldsymbol{\lambda}$$

subjected to

a) $[\mathbf{E}]\,\mathbf{q} - [\mathbf{N}]\,\boldsymbol{\lambda} = \mathbf{0}$,

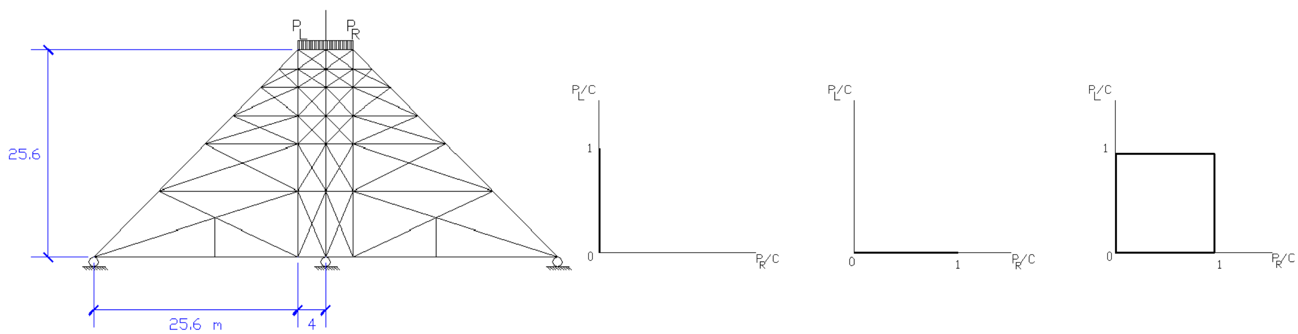
b) $\mathbf{M}^T \boldsymbol{\lambda} = 1$

c) $\boldsymbol{\lambda} \geq \mathbf{0}$ (5)

It was proved, by Maier [3] that the linear program in Eq(5) represents Koiter's kinematic theorem, if q and λ are the vectors of nodal displacements and plastic multipliers respectively. This theorem states that, a body will not shake down, i.e., it will fail by cyclic plastic deformation, if for any admissible plastic strain rate, and any generalized load combination, the rate of external work over one cycle exceeds the plastic energy dissipation rate over that cycle. Constraints identified at Eq(5a) represent a kinematically admissible plastic strain rate cycle, Koiter [2], while constraint Eq(5b) represents the maximum positive external work condition, and the objective function represents the internal plastic energy dissipation.Equation (5) is a quasi-static generalization of the linearlized form of the kinematic theorem of limit analysis, Anderheggen [12].

3. NUMERICAL RESULTS:

The problem considered is a symmetric plane strain trapezoidal embankment with 45° side slopes subjected to uniform repeated load as shown in Fig (1). Either the right lane is loaded, or the left lane is loaded or both lanes are loaded, while the side slopes are always stress free. Elastic, shakedown and limit loads are obtained for each case of loading. The effect of internal friction, φ, on the elastic, shakedown and limit capacities of the embankment is investigated.



a) Left Lane Loading Program

b) Right Lane Loading Program

c) The Two Lanes Loading Program

Fig (1) Finite Element Discretization for the Field Problem and Loading Programs, (PL and PR are traction loading and C is the cohesion).

The material of the embankment is modeled as an elastic-perfectly plastic material obeying piecewise linearized Mohr-Coulomb yield criteria with its associated flow rule, Fig (2). Four different cases of internal friction are considered, φ = 0, 10° ,20° and 30°, while the elastic properties are kept constants, Young's modulus is 2000 t/m2 and Poisson's ratio is 0.30.

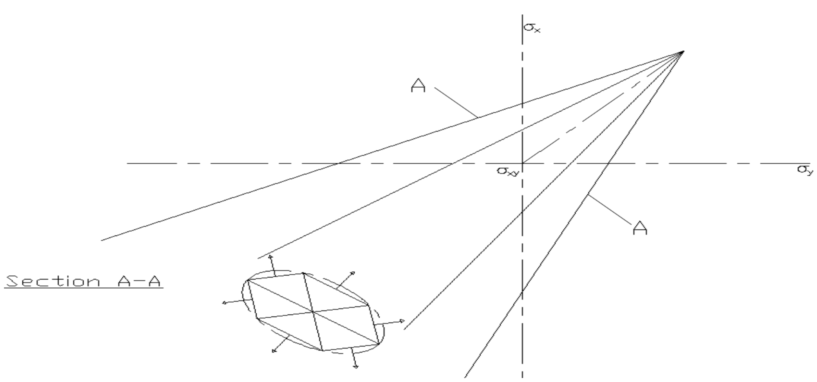


Fig (2): Piecewise Linearized Mohr-Coulomb Yield Surface in Plane Strain, Anderheggen [12]

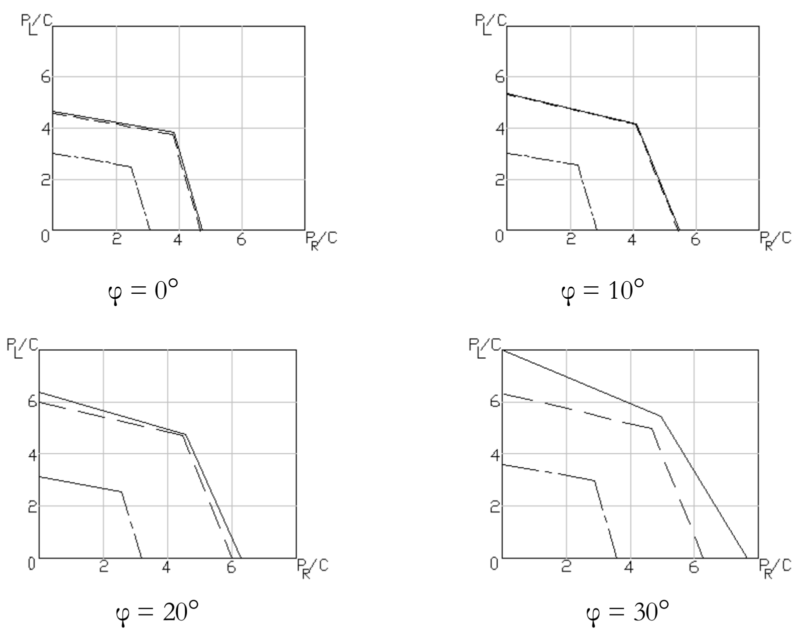
Also, Fig (1) illustrates the finite element discretization of the embankment into 92 elements and 57 nodes. The static formulation is adopted for both limit and shakedown analyses, which requires 1105 variables and 652 constraints for each analysis. Table (1) and Figs (3, 4) illustrate the results of the analysis. This same mesh was used in the verification of the program [8] and showed excellent agreement with analytical solutions. For example, when $\varphi = 0$ and proportional loading, the limit load shown in Table (1) is in excellent agreement with the analytical upper bound solution Pu given in [14] in terms of the cohesion, C, as;

$$Pu = 2C\,(1 + \pi/4) \tag{6}$$

Also, it can be seen that all load factors for all analyses (elastic, shakedown and limit loads) increase as the angle of the internal friction increases. Also, the limit load factor not only has the highest magnitude, but it also has the highest gradient with respect to φ. Thus, the gap between limit and shakedown factors increases as φ increases. Consequently roads which are generally subjected to repeated development of plastic strains should therefore, by definition, be designed according to a shakedown criterion, such as that illustrated here by the curves S2 in Fig (4). Although the limit analysis (curves L1, L2 in Fig (4)) provides almost the same shakedown factors at small angles of friction, it provides unsafe estimate of the collapse loads at higher angles of friction. On the other hand, the elastic analysis curves E1, E2 in Fig (4)), can be considered as an over conservative design approach, and will result in a waste of the material strength, as failure occurs at loads much higher than those provided by the elastic analysis.

	Elastic Analysis		Shakedown Analysis		Limit Analysis	
φ°	One Lane	Two Lanes	One Lane	Two Lanes	One lane	Two Lanes
0	2.59247	2.27605	4.45807	3.49815	4.49663	3.51886
10	2.89441	2.53953	5.19998	4.02135	5.27623	4.02148
20	3.12564	2.833353	5.99085	4.68559	6.31269	4.83231
30	3.35638	3.10221	6.4121	5.49339	7.82657	5.8809

Table (1) : Elastic, Shakedown Limit Load Factors Due to Different Loading Programs



____: Limit Analysis -- . --: Elastic Analysis __ __ : Shakedown Analysis

Fig (3): Elastic, Shakedown and Limit Domains for the field Problem at different angles of internal friction

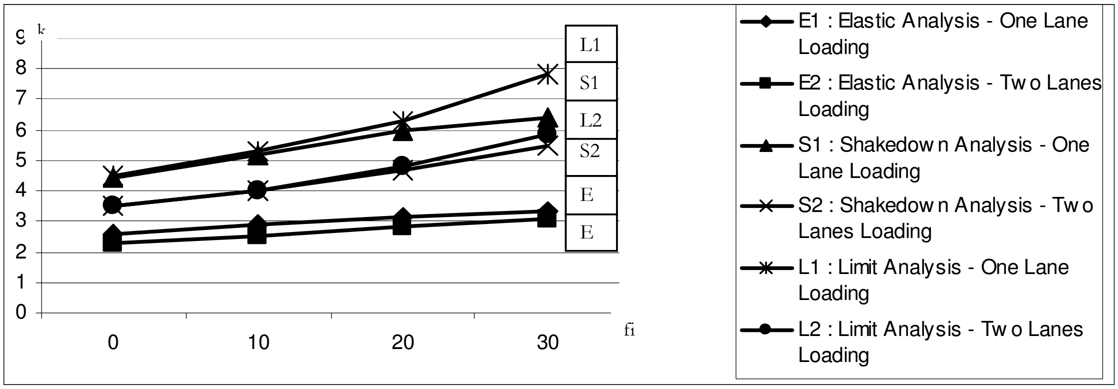


Fig. (4): Elastic, Shakedown and Limit Load Factors Due to Different Loading Programs and at Different Angles of Internal Friction.

4. CONCLUSION:

Field applications to an embankment problem subjected to cyclic loading indicate that a tremendous saving in the material strength could be achieved if these embankments were designed according to a collapse criterion rather than an elastic one. This collapse criterion should be manifested by a shakedown analysis as failure occurred at loads below the limit values especially at higher angles of internal friction. On the other hand, the elastic analysis usually used in road design is an over conservative design approach, as failure occurs at loads much higher than those provided by the elastic analysis.

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